

# MATH PART-I ( PAPER-I, THEORY OF EQUATION)

## DESCARTE'S RULE OF SIGN

Q :- Apply Descartes's rule of signs to find the nature of the roots of the equation  $x^4 + 15x^2 + 7x - 11 = 0$

Ans:- Let  $f(x) = x^4 + 15x^2 + 7x - 11$

Obviously the equation is of even degree whose last term is negative and the co-efficient of first term is positive. Hence it will have two real roots, one +ve and the other -ve. Therefore, the other two roots are imaginary. Given equation is an equation of 4th degree. Hence it must have 4 roots. Therefore, the other two roots are imaginary.

—x—x—

Q :- Using Descartes's rule of sign, find the nature of the roots of the equation  $x^4 + 15x^2 + 7x - 11 = 0$

Ans:- Let  $f(x) = x^4 + 15x^2 + 7x - 11 = 0$

Firstly we observe that the given Polynomial equation is of 4th degree. Hence it will have four roots, real or imaginary. Again, we have  $f(x) = x^4 + 15x^2 + 7x - 11$

Here the signs of the terms of the Polynomial are  
+ + + -

Since there is one change of sign, the given equation  $f(x) = 0$  has only one positive root.

Again  $f(-x) = x^4 + 15x^2 - 7x - 11$

Thus  $f(-x)$  has only one change of sign.

Hence  $f(x) = 0$  will have only one negative root.

Now  $f(-\infty) = +ve$ ,  $f(0) = -ve$ ,  $f(\infty) = +ve$

Hence combining ① and ②, we find that the equation has one positive root and one negative root.

—x—

Q. 1: Show that the quintic equation  $x^5 + x^3 - 8x - 5 = 0$  cannot have more than three real roots and prove that it must have three real roots. (20)

Ans:- We have  $f(x) = x^5 + x^3 - 8x - 5 = 0$

Since the equation  $f(x) = 0$  has only one change of sign it must have one positive root and no more.

Again, we have  $f(-x) = -x^5 - x^3 + 8x - 5 = 0$

Obviously the equation  $f(-x) = 0$  has two changes of sign and as such the equation  $f(-x) = 0$  has only two positive real roots. That is  $f(x) = 0$  has at the most two negative roots.

Thus in all  $f(x) = 0$  will have three real roots, one +ve and two -ve obviously, we have from (1)

$$f(0) = -ve$$

$$f(-\infty) = -ve$$

$$f(\infty) = +ve$$

$$f(-1) = +ve$$

Since  $f(0)$  and  $f(-\infty)$  are of the same sign and as such there exists either no real roots or an even number of roots of  $f(x) = 0$  lies between 0 and  $-\infty$ . Further, we have  $f(0)$  and  $f(-1)$  to be of opposite signs, then at least one or an odd number of real roots of the equation  $f(x) = 0$  lies between 0 and -1,

We also have,

$$f(1) = -11 \quad (-ve)$$

$$f(2) = 32 + 8 - 16 - 5 = 19 \quad (+ve)$$

$$f(-2) = -32 - 8 + 16 - 5 = -29 \quad (-ve)$$

Since  $f(1)$  and  $f(2)$  are of contrary signs. Hence at least one or odd number of real roots of the equation  $f(x) = 0$  will lie between 1 and 2. Also  $f(-2)$  and  $f(-1)$  are of contrary (opposite) signs one root will lie between -2 and -1. Thus, we conclude that the given equ.  $f(x) = 0$  must have three real roots and hence the remaining two roots of the equation

Q. i- Prove that the equation  $x^5 - x + 16 = 0$  has two pairs of complex roots. (3)

Ans. - Let  $f(x) \equiv x^5 - x + 16 = 0$

Obviously, the given equation  $f(x) = 0$  is an equation of odd degree and the last term is positive, hence it will have at least one real root.

Also  $f(x)$  has two changes of sign. Hence there cannot be more than two positive real roots of  $f(x) = 0$ .

Again,  $f(-x) = -x^5 + x + 16 = 0$  or,  $x^5 - x - 16 = 0$

Thus  $f(-x)$  has only one change of sign and such as  $f(x) = 0$  cannot have more than one negative root.

Thus we conclude that the equation  $f(x) = 0$  have at least two complex roots. Also from (1), we have

$$f(0) = +ve, f(+\infty) = +ve, f(-\infty) = -ve.$$

Since  $f(0)$  and  $f(+\infty)$  are of same sign, then either no real root or an even number of roots of  $f(x) = 0$  lies between 0 and  $+\infty$ . Further, there exists no real quantities which, substituted for  $x$ , makes  $f(x) = 0$ , hence  $f(x)$  must be positive for every real value of  $x$ .

It follows that the equation  $f(x) = 0$ , cannot have a positive root. Further, we have  $f(0) = +ve$  and  $f(-\infty) = -ve$  and hence one negative root lies between 0 and  $-\infty$ .

Thus from above discussion, we can conclude that the given equation  $x^5 - x + 16 = 0$  has only one negative real root which lies between 0 and  $-\infty$ . But the given equation is a quintic i.e. of fifth degree. We infer that the remaining four roots must be complex further, as the complex roots always occur in conjugate pairs, we conclude that the given equation has two pairs of complex roots.

————— x ————— x —————

Q :- Show that the equation  $\frac{A^2}{x-a} + \frac{B^2}{x-b} + \frac{C^2}{x-c} + \dots + \frac{H^2}{x-h} = k$  ④  
 has no imaginary roots, where  $A, B, C, \dots, H$  and  $a, b, c, \dots, h$   
 all are real.

Ans:- 
$$\frac{A^2}{x-a} + \frac{B^2}{x-b} + \frac{C^2}{x-c} + \dots + \frac{H^2}{x-h} = k \quad \text{--- (1)}$$

This problem is in general form.

If possible let  $\alpha - i\beta$  be an imaginary root of the given equation, then  $\alpha + i\beta$  will also be its root.

On substituting these values for  $x$  in the given equation, and arranging the terms,

$$\frac{A^2}{(\alpha - a) + i\beta} + \frac{B^2}{(\alpha - b) + i\beta} + \frac{C^2}{(\alpha - c) + i\beta} + \dots + \frac{H^2}{(\alpha - h) + i\beta} = k \quad \text{--- (2)}$$

and 
$$\frac{A^2}{(\alpha - a) - i\beta} + \frac{B^2}{(\alpha - b) - i\beta} + \frac{C^2}{(\alpha - c) - i\beta} + \dots + \frac{H^2}{(\alpha - h) - i\beta} = k \quad \text{--- (3)}$$

Subtracting (2) from (3).

$$\frac{A^2 \{(\alpha - a) - i\beta - (\alpha - a) + i\beta\}}{\{(\alpha - a) + i\beta\}\{(\alpha - a) - i\beta\}} + \frac{B^2 \{(\alpha - b) - i\beta - (\alpha - b) + i\beta\}}{\{(\alpha - b) + i\beta\}\{(\alpha - b) - i\beta\}} + \frac{C^2 \{(\alpha - c) - i\beta - (\alpha - c) + i\beta\}}{\{(\alpha - c) + i\beta\}\{(\alpha - c) - i\beta\}} + \dots + \frac{H^2 \{(\alpha - h) - i\beta - (\alpha - h) + i\beta\}}{\{(\alpha - h) + i\beta\}\{(\alpha - h) - i\beta\}} = 0$$

$$\therefore -2i\beta \left[ \frac{A^2}{(\alpha - a)^2 + \beta^2} + \frac{B^2}{(\alpha - b)^2 + \beta^2} + \frac{C^2}{(\alpha - c)^2 + \beta^2} + \dots + \frac{H^2}{(\alpha - h)^2 + \beta^2} \right] = 0$$

Obviously, the expression within the bracket is the sum of positive quantities and as such, it cannot be zero.

Hence  $2i\beta = 0 \Rightarrow \beta = 0$ , Thus there is no imaginary root of the given equation (1) and the roots are all real.

— x ————— x —————

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Q: Two roots of the equation  $x^5 - x^4 + 8x^2 - 9x - 15 = 0$  are  $-\sqrt{3}$  and  $1+2i$ . Find the other roots of the equation.

Ans: Let us suppose that  $1+2i$  is a root of  $x^5 - x^4 + 8x^2 - 9x - 15 = 0$ . Then  $1-2i$  is also a root of the given equation. Similarly, if  $-\sqrt{3}$  is a root of the given equation, then  $+\sqrt{3}$  will be the other root. As imaginary and irrational roots always occur in pairs. The product of the factors corresponding to these roots is given by.

$$(x+\sqrt{3})(x-\sqrt{3})(x-1+2i)(x-1-2i) = (x^2-3)\{(x-1)^2+4\}$$

$$= (x^2-3)(x^2-2x+1+4)$$

$$= (x^2-3)(x^2-2x+5)$$

$$= x^4 - 2x^3 + 5x^2 - 3x^2 + 6x - 15$$

$$= x^4 - 2x^3 + 2x^2 + 6x - 15$$

Now we find that it divides the L.H.S. of the given equation exactly without remainder giving the quotient as  $x+1$ , which corresponds to the other roots. Hence our assumption is correct and the other root be given by  $x+1=0$  i.e.  $x=-1$ .

Hence the remaining roots  $\sqrt{3}$ ,  $1-2i$  and  $-1$ .

